Multivariable Calculus: Problem Set 1A

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Instructions: You are encouraged to discuss general strategies to approach the questions on this problem set with your classmates, but you must work and write up your solutions to the problems entirely on your own. Of course, you are always welcome to meet with me to talk about any question you are having difficulty with. Please pay attention to making your solutions as clear as possible for the reader; mathematical communication is an important skill that you will develop in this course.

Problem 1

Consider the equation $x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$, and let $K = G^2 + H^2 + I^2 - 4J$. Before reading the rest of this problem, take a moment and try to figure out what figure the equation represents.

Prove that the equation represents a sphere if $K > 0$, a point if $K = 0$, and has no graph if $K < 0$. In the case where $K > 0$ find the center and radius of the sphere. In the case where $K > 0$, how close and how far does the sphere get from the origin? Be sure to explain why your answers represent the minimum and maximum distance from the origin.

Hint: There may be two cases you need to consider.

Problem 2

Using only your knowledge of 2-dimensional geometry, prove that the distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The distance formula in 3-dimensions is intuitive, with our knowledge from 2-dimensions. But for this problem, I want you to think about explaining *why* this formula holds true.

Problem 3-1

A vector $\vec{w}$ is a *linear combination* of the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ if $\vec{w}$ can be expressed as $\vec{w} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$, where c_1, c_2, and c_3 as scalars.

(a) Find scalars c_1, c_2, and c_3 to express $< -1, 1, 5 >$ as a linear combination of $\vec{v_1} =< 1, 0, 1 >$, $\vec{v_2} =< 3, 2, 0 >$, and $\vec{v_3} =< 0, 1, 1 >$.

(b) Show that the vector $2\vec{i} + \vec{j} - \vec{k}$ cannot be expressed as a linear combination of $\vec{v_1} = \vec{i} - \vec{j}$, $\vec{v_2} = 3\vec{i} + \vec{k}$, and $\vec{v_3} = 4\vec{i} - \vec{j} + \vec{k}$

Note: What we're encountering here is the beginnings of linear algebra. A fundamental question in linear algebra is this: given a set of vectors, what are all vectors that can you form from them, if you only can add, subtract, and multiply them by a scalar? As you showed, you can't always get all possible vectors from a linear combination. It depends entirely on the vectors that we're using to form the linear combinations. If the vectors we were using were $< 1, 0, 0 >$, $< 0, 1, 0 >$, and $< 0, 0, 1 >$, however, we could get any vector: $< a, b, c >= a < 1, 0, 0 > +b < 0, 1, 0 > +c < 0, 0, 1 >$. So here's something to mull over: if you want to be able to get any vector out of a linear combination, what constraints would we need to put on the original set of vectors?

Problem 3-2

Show that if $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ are mutually orthogonal nonzero vectors in 3-space, and if a vector $\vec{v}$ in 3-space is expressed as

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

then the scalars c_1, c_2, and c_3 are given by the formulas

$$c_i = \frac{\vec{v} \cdot \vec{v_i}}{\|\vec{v_i}\|^2} \text{ for } i = 1, 2, 3$$

Problem 4

Show that in 3-space the distance d from a point P to the line L through the points A and B can be expressed as

$$d = \frac{\|\vec{AP} \times \vec{AB}\|}{\|\vec{AB}\|}$$

Problem 5

Use your graphing calculators to approximate the minimum area of a triangle if two of its vertices are $(2, -1, 0)$ and $(3, 2, 2)$ and its third vertex is on the curve $y = \ln x$ in the xy-plane.

Be sure to show all algebraic steps, in addition to drawing a rough sketch of your graph, the equation you graphed, and the window you used.