Kepler's Laws of Planetary Motion

December 1, 2008

Section 1: Introduction, or What We're Trying to Figure Out

Let us tell you a story about two celestial bodies – called the sun and the earth. Have you ever wondered why the earth travels around the sun in an elliptical fashion, instead of a nice, perfect circle? Our friend Johannes Kepler figured it out. We're going to amaze you by proving our friend's theory, using only multivariable calculus and the laws of forces and gravitation handed down by Newton, Sir Isaac Newton.

By the end of this paper, you will come to understand that depending on the initial conditions, a planet will travel around the sun in a conic section (a circle, ellipse, parabola, or hyperbola). Unfortunately, we don't have enough information about the initial conditions of the solar system to show the earth moves in an ellipse.[1] We will, however, show that it is mathematically *possible* that the earth moves in an ellipse. Astronomical observations by our old pal Kepler and others since him have proven that indeed the earth moves in an ellipse around the sun, with the sun at a focus.

Section 2: Playing God, or Setting Initial Conditions

We are going to play God and set up the universe. We will place a sun at the origin of this 3-D universe, and this sun will have mass M. We will also place Earth, with mass m, a distance r from the sun at the start of the universe ($t = 0$). We will say that this distance will be minimal when the universe begins.

We will define the position and velocity of the earth to be $\vec{r}$ and $\vec{v}$, and state that at the beginning of the universe, these values are $\vec{r_0}$ and $\vec{v_0}$ (and are non-zero).[2]

For simplicity's sake, we will assume that $\vec{r_0}$ and $\vec{v_0}$ are perpendicular to each other.

As always, we know $\vec{v} = \frac{d}{dt}[\vec{r}]$ (the rate of change of the position over time will be the velocity). Similarly, $\vec{a} = \frac{d}{dt}[\vec{v}]$.

Section 3: Gravitational Attraction

PART I: Equal and Opposite, or Understanding $\vec{r}$ and $\vec{a}$

From Newton's second law ($\vec{F} = m\vec{a}$), we know that the acceleration of earth ($\vec{a}$) is moving in the same direction as the force exerted on earth by the sun ($\vec{F}$). See the attached diagram to see the forces exerted on the sun by the earth, and vice versa.

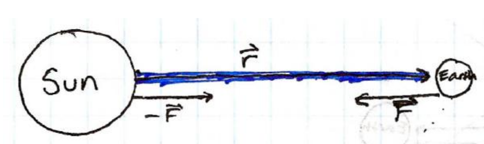

Figure 1:

[1] This is a math class after all, not an astronomy class.

[2] If the initial velocity was zero, you would not have to read this paper, because the earth would crash horrifically into the sun. If the initial position was zero, the earth would be inside the sun. Eesh!

As you can see above, the direction of the acceleration of the earth, $\vec{a}$, goes in the opposite direction of $\vec{r}$.

Since we know that $\vec{r}$ and $\vec{a}$ are in opposite directions, we know that

$$\vec{r} \times \vec{a} = \vec{0}$$

Now we've proven that $\vec{r}$ and $\vec{a}$ are opposite. Let's keep that in mind as we move on.

PART II: How the Earth moves on a Plane, or Defining Vector $\vec{b}$

To prove that the earth travels around the sun on a plane, we must show that $\vec{r} \times \vec{v}$ is a constant. If it is indeed a constant, we know that the normal vector is the same at any time t. This is the definition of a plane (that the direction of the normal vector is the same at all points). To prove $\vec{r} \times \vec{v}$ that is a constant at every moment in time, we will show that $\frac{d}{dt}[\vec{r} \times \vec{v}] = 0$. By the chain rule:

$$\frac{d}{dt}[\vec{r} \times \vec{v}] = \vec{r} \times \frac{d}{dt}[\vec{v}] + \frac{d}{dt}[\vec{r}] \times \vec{v}$$

Since we know that $\frac{d}{dt}[\vec{v}] = \vec{a}$ and $\frac{d}{dt}[\vec{r}] = \vec{v}$, we find we can simplify the equation above to be:

$$\frac{d}{dt}[\vec{r} \times \vec{v}] = \vec{r} \times \vec{a} + \vec{v} \times \vec{v}$$

Since we showed in Part I that $\vec{r} \times \vec{a} = \vec{0}$, and since $\vec{v} \times \vec{v} = \vec{0}$ because both vectors are in the same direction, we can finally conclude what we set out to show:

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$$\frac{d}{dt}[\vec{r} \times \vec{v}] = \vec{0}$$

From this, we know that $\vec{r} \times \vec{v}$ is unchanging over time – hence it will be a constant. We will name this vector $\vec{b}$.

$$\vec{r} \times \vec{v} = \vec{b}$$

Note that $\vec{r}$ and $\vec{v}$ must change over time to keep $\vec{b}$ constant. When one has a larger magnitude, the other must have a smaller magnitude, and vice versa.

We have also shown what we said we wanted to show, that the direction of the normal vector remains unchanging over time, so we can conclude that earth moves in a plane. We don't yet know it moves in an ellipse, or even around the sun!

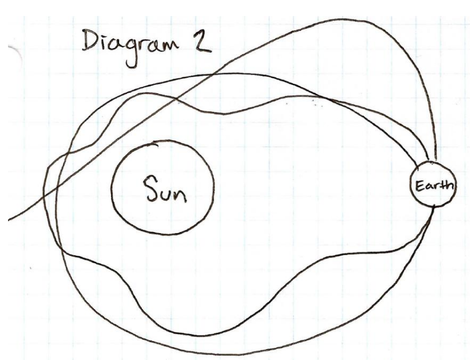


Figure 2:

PART III: Gravitation, or Relating $\vec{a}$ and $\vec{r}$

In this section, we are going to use the Newton's Law of Universal Gravitation and Newton's Second Law to relate acceleration, $\vec{a}$, to position, our vector $\vec{r}$.

Newton's Law of Universal Gravitation states that

$$\left\|\vec{F}\right\| = \frac{GMm}{r^2}$$

where G represents the universal gravitation constant[3], M is the mass of the sun, m is the mass of earth, and r is the distance from the sun to the earth (which changes over time).[4]

To write $\vec{F}$as a vector quantity, we need both a magnitude and direction. Clearly Newton's Law of Gravitation gives us the magnitude. The direction of the force must point from earth to the sun, which is in the opposite direction of the position vector ($\vec{r}$). Therefore we can write

$$\vec{F} = \frac{GMm}{r^2}\left(-\frac{\vec{r}}{\|\vec{r}\|}\right)$$

The second factor in the equation above gives us a unit vector in the direction of $\vec{F}$. Recognizing that $\|\vec{r}\| = r$, we can rewrite the equation above

$$\vec{F} = -\frac{GMm}{r^3}\vec{r}$$

Using Newton's Second Law ($\vec{F} = m\vec{a}$), we can substitute in for $\vec{F}$ and divide by m to get

$$\vec{a} = -\frac{GM}{r^3}\vec{r}$$

Now we have accomplished what we set out to do!

Section 4: Understanding $\vec{b}$ in Rectangular and Cylindrical Coordinates

PART I: Rectangular Coordinates

We know $\vec{b}$ – the vector normal to the plane on which the earth moves – is a constant vector at all times. We want to find the magnitude and direction of this vector. Since it is constant for all t, we can find the value of $\vec{b}$ at the starting point of the universe ($t = 0$). Recall from our initial conditions that $\vec{r}(0) = \vec{r}_0$ and $\vec{v}(0) = \vec{v}_0$, and that $\vec{r}_0$ and $\vec{v}_0$ are perpendicular.

We already showed that the motion of the planet will lie on a plane. Let us call this plane the $x - y$ plane – with the sun at the origin and the starting position of the planet on the positive x-axis. See the diagram below.

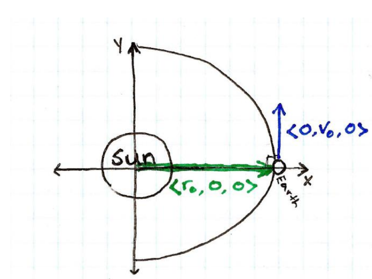


Figure 3:

[3] $G = 6.67 \times 10^{-9} Nm^2/kg^2$

[4] r changes over time, so we could in theory write it as $r(t)$. However, we assume that our astute readership will keep this in mind for the rest of the paper.

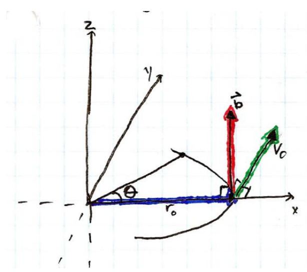
This allows us to say $\vec{r_0} =< r_0, 0, 0 >$ and $\vec{v_0} =< 0, v_0, 0 >$

Since $\vec{r} \times \vec{v} = \vec{b}$ is constant at all times t, we say $\vec{r_0} \times \vec{v_0} = \vec{b}$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_0 & 0 & 0 \\ 0 & v_0 & 0 \end{vmatrix} =< 0, 0, r_0 v_0 >$$

So $\vec{b}$ is a vector pointing in the z-direction (coming out of the $x - y$ plane) with magnitude $r_0 v_0$.

Figure 4:



PART II: Cylindrical Coordinates

Again, we want to find a different way to represent $\vec{b}$ (this time in cylindrical coordinates), so that ultimately – take a deep breath, trust us here – we will eventually use these two ways of looking at $\vec{b}$ to find a second equation which doesn't change over time.

Since $\vec{b}$ is defined by $\vec{r}$ and $\vec{v}$, we will convert $\vec{r}$ into cylindrical coordinates using the standard conversion. Let's call $\vec{r} =< x, y, 0 >$. As always, recall that we write$\|\vec{r}\| = r$.[5] The conversion factors are

$$x = r \cos\theta$$

$$y = r \sin\theta$$

So cylindrically, we can express $\vec{r}$ as

$$\vec{r} =< r\cos\theta, r\sin\theta, 0 >= r < \cos\theta, \sin\theta, 0 >$$

We will designate the important second term $\vec{u} =< \cos\theta, \sin\theta, 0 >$ which gives us a direction (with a unit magnitude). The r gives us the magnitude. Together they fully express the position vector $\vec{r}$.

Therefore we can rewrite $\vec{r}$ as $r\vec{u}$.

Now we need to express $\vec{v}$ in cylindrical coordinates.

$$\vec{v} = \frac{d}{dt}[\vec{r}] = \frac{d}{dt}[r\vec{u}] = \vec{u}\frac{d}{dt}[r] + r\frac{d}{dt}[\vec{u}]$$

We needed to use the chain rule because both r and $\vec{u}$ are functions of time. (Remember r is not a vector, but the distance from the earth to the sun may change over time.)

Now that we've found both $\vec{r}$ and $\vec{v}$ in cylindrical coordinates, let's express $\vec{b}$ in cylindrical coordinates by calculating the cross product:

[5] $r = \sqrt{x^2 + y^2 + 0^2}$

$$\vec{b} = \vec{r} \times \vec{v}$$

$$\vec{b} = (r\vec{u}) \times (\vec{u}\frac{d}{dt}[r] + r\frac{d}{dt}[\vec{u}])$$

Now we are going to show step-by-step-by-step how we solve this ugly beast.

Using the distribution properties associated with cross products:

$$\vec{b} = (r\vec{u}) \times (\vec{u}\frac{d}{dt}[r]) + (r\vec{u}) \times (r\frac{d}{dt}[\vec{u}])$$

Taking out the scalars:

$$\vec{b} = r\frac{d}{dt}[r]\,[\vec{u} \times \vec{u}] + r^2\left[\vec{u} \times \frac{d}{dt}[\vec{u}]\right]$$

Since a vector crossed with itself is the zero vector, the first term drops out and we're left with:

$$\vec{b} = r^2\left[\vec{u} \times \frac{d}{dt}[\vec{u}]\right]$$

Now we will explicitly show you how to calculate the cross product. To do this, we need to take the derivative of $\vec{u}$. Since $\vec{u} =< \cos\theta, \sin\theta, 0 >$, we know that $\frac{d}{dt}\vec{u} =< -\sin\theta\frac{d\theta}{dt}, \cos\theta\frac{d\theta}{dt}, 0 >=< -\sin\theta, \cos\theta, 0 > \frac{d\theta}{dt}$. Applying these to the cross product

$$\vec{b} = r^2\left[\vec{u} \times \frac{d}{dt}[\vec{u}]\right]$$

$$\vec{b} = r^2\left[< \cos\theta, \sin\theta, 0 > \times (< -\sin\theta, \cos\theta, 0 > \frac{d\theta}{dt})\right]$$

Again, factoring out the scalar:

$$\vec{b} = r^2\frac{d\theta}{dt}\,[< \cos\theta, \sin\theta, 0 > \times < -\sin\theta, \cos\theta, 0 >]$$

$$\vec{b} = r^2\frac{d\theta}{dt}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$\vec{b} = r^2\frac{d\theta}{dt} < 0, 0, 1 >=< 0, 0, r^2\frac{d\theta}{dt} >$$

The ugly beast has thus been vanquished, with great aplomb.

Section 5: The Scalar Triple Product, Oh My!

We are going to find the scalar triple product to show that vectors $\vec{r}$, $\vec{v}$, and $\vec{b}$ form a parallelapiped[6] which has an unchanging volume over time. In this section, we will express the volume in two different ways and then set them equal to each other. Using this equality, we will derive a formula for r – the distance from the earth to the sun. Notice, and we will discuss this later, that this method will only yield us a scalar. We won't be able to find a formula for $\vec{r}$.

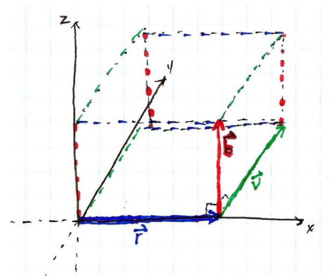


Figure 5:

PART I: A Constant Scalar Triple Product

The volume of the parallelepiped is $(\vec{r} \times \vec{v}) \bullet \vec{b}$. Recall that we defined $\vec{b} = \vec{r} \times \vec{v}$. With simple substitution, we can write the volume as:

$$Vol = \vec{b} \bullet \vec{b}$$

From Section 4, Part I, we know that $\vec{b} =< 0, 0, r_0 v_0 >$, so the volume is now

$$Vol = r_0^2 v_0^2$$

This shows that the volume of the parallelepiped is a constant over time, dependent on the initial position and initial velocity.

PART II: A Time-Dependent Scalar Triple Product

Again, we are going to find the volume of this parallelepiped in a second way. Recall the definition of a scalar triple product is:

$$Vol = \vec{r} \bullet (\vec{v} \times \vec{b})$$

Clearly what we don't know is $\vec{v} \times \vec{b}$. Let's figure it out! We don't yet have an equation for vector $\vec{v}$, so we're going to create an equation in terms of what we *do* know ($\vec{a}$ and $\vec{b}$).

$$\frac{d}{dt}[\vec{v} \times \vec{b}] = \frac{d}{dt}[\vec{v}] \times \vec{b} + \vec{v} \times \frac{d}{dt}[\vec{b}]$$

Since $\vec{b}$ is unchanging over time, $\frac{d}{dt}[\vec{b}] = \vec{0}$, and $\frac{d}{dt}[\vec{v}] = \vec{a}$. Hence we have

$$\frac{d}{dt}[\vec{v} \times \vec{b}] = \vec{a} \times \vec{b}$$

[6]Since $\vec{r}$, $\vec{v}$, and $\vec{b}$ are all orthogonal to each other, the scalar triple product is more specifically the volume of a box (a rectangular prism).

Finally, we can solve this cross product the standard way. However, we need everything to be in cylindrical coordinates. We have $\vec{b}$ in cylindrical coordinates, but we still need to convert $\vec{a}$.

$$\vec{a} = -\frac{GM}{r^3}\vec{r} = -\frac{GM}{r^3}r\vec{u} = -\frac{GM}{r^2}\vec{u} = -\frac{GM}{r^2} < \cos\theta, \sin\theta, 0 >$$

In these, notice we substituted $\vec{r} = r\vec{u}$, and we wrote all the terms of $\vec{u}$ out to take the cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{GM}{r^2}\cos\theta & -\frac{GM}{r^2}\sin\theta & 0 \\ 0 & 0 & r^2\frac{d\theta}{dt} \end{vmatrix}$$

$$\vec{a} \times \vec{b} =< -GM\sin\theta\frac{d\theta}{dt}, GM\cos\theta\frac{d\theta}{dt}, 0 >$$

$$\vec{a} \times \vec{b} = GM < -\sin\theta\frac{d\theta}{dt}, \cos\theta\frac{d\theta}{dt}, 0 >$$

Surprise! Notice that the vector above is simply $\frac{d}{dt}[\vec{u}]$. Thus we can finally write

$$\vec{a} \times \vec{b} = GM\frac{d}{dt}[\vec{u}]$$

Whoa, that was a lot of work. Let's remind ourselves what we're trying to do here with all these equations! We're trying to find the volume of our parallelepiped, $Vol = \vec{r} \bullet (\vec{v} \times \vec{b})$. We couldn't find $\vec{v} \times \vec{b}$, but we were able to find $\vec{a} \times \vec{b}$. But hark! We can integrate $\vec{a} \times \vec{b}$ to get $\vec{v} \times \vec{b}$, because in Section 5, Part II, we found $\frac{d}{dt}[\vec{v} \times \vec{b}] = \vec{a} \times \vec{b}$.

$$\int \frac{d}{dt}[\vec{v} \times \vec{b}]dt = \int \vec{a} \times \vec{b}dt = \int GM\frac{d}{dt}[\vec{u}]dt = GM\vec{u} + \vec{C}$$

$$\vec{v} \times \vec{b} = GM\vec{u} + \vec{C}$$

To find $\vec{C}$, we look at the state of the system at $t = 0$. The values of $\vec{v}$ and $\vec{b}$ at $t = 0$ are $< 0, v_0, 0 >$ and $< 0, 0, r_0 v_0 >$. The cross product of these initial vectors is $< r_0 v_0^2, 0, 0 >$. At $t = 0$, we know $GM\vec{u} + \vec{C} =< GM, 0, 0 > +\vec{C}$. Hence, $\vec{C} =< r_0 v_0^2 - GM, 0, 0 >$. Finally

$$\vec{v} \times \vec{b} = GM\vec{u} + < r_0 v_0^2 - GM, 0, 0 >$$

And finally, the parallelepiped! Since we now have all the necessary tools to calculate the scalar triple product, let's make like Nike and just do it!

$$Vol = \vec{r} \bullet (\vec{v} \times \vec{b})$$

$$Vol = \vec{r} \bullet (GM\vec{u} + < r_0 v_0^2 - GM, 0, 0 >) = \vec{r} \bullet GM\vec{u} + \vec{r} \bullet < r_0 v_0^2 - GM, 0, 0 >$$

Writing all the vectors out:

$$Vol =< r\cos\theta, r\sin\theta, 0 > \bullet GM < \cos\theta, \sin\theta, 0 > + < r\cos\theta, r\sin\theta, 0 > \bullet < r_0 v_0^2 - GM, 0, 0 >$$
$$Vol = r < \cos\theta, \sin\theta, 0 > \bullet GM < \cos\theta, \sin\theta, 0 > + r < \cos\theta, \sin\theta, 0 > \bullet < r_0 v_0^2 - GM, 0, 0 >$$
$$Vol = GMr < \cos\theta, \sin\theta, 0 > \bullet < \cos\theta, \sin\theta, 0 > + r < \cos\theta, \sin\theta, 0 > \bullet < r_0 v_0^2 - GM, 0, 0 >$$
$$Vol = GMr + r\cos\theta(r_0 v_0^2 - GM)$$
$$Vol = r(GM + \cos\theta(r_0 v_0^2 - GM))$$

(Note in the penultimate step, we took the dot product $< \cos\theta, \sin\theta, 0 > \bullet < \cos\theta, \sin\theta, 0 >= \cos^2\theta + \sin^2\theta$. That sums to 1 using our favorite trigonometric identity.)

PART III: Settin' 'em into Place

Finally, notice we derived the volume in two different ways, in Part I and Part II of this section, in the hopes to find an equation for r. We can equate them, and see magical things happen.

$$r_0^2 v_0^2 = r(GM + \cos\theta(r_0 v_0^2 - GM))$$

$$r = \frac{r_0^2 v_0^2}{GM + \cos\theta(r_0 v_0^2 - GM)}$$

Look at this ugly mess. Let's make it a little nicer. Let's rewite by dividing the numerator and denominator by GM:

$$r = \frac{\frac{r_0^2 v_0^2}{GM}}{1 + \cos\theta(\frac{r_0 v_0^2}{GM} - 1)}$$

Ergo, magic.

Often times, this equation is written as

$$r = \frac{k}{1 + e\cos\theta}$$

where the constants are defined by $k = \frac{r_0^2 v_0^2}{GM}$ and $e = \frac{r_0 v_0^2}{GM} - 1$.