21st Century Mathematics with Justin Lanier - Week 1

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I'm working on these problems not having read the chapter in the book about these fractions.

Part (a) Give an example of exactly three fractions that when you add them together you get 1.

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$$

Can you also come up with an example where the fractions are all different?

$$\frac{1}{4}+\frac{-1}{4}+\frac{1}{1}$$

What about three fractions that add up to 1 and all of their denominators are different from each other?

$$\frac{1}{6}+\frac{1}{4}+\frac{1}{2}$$

Part (b) A unit fraction is a fraction with a numerator of 1 and denominator that is a whole number, 2 or more. For example, $\frac{1}{3}$ or $\frac{1}{15}$ are unit fractions. (Sometimes $\frac{1}{1}$ is considered a unit fraction, sometimes not. For these problems, we will not count it as a unit fraction.

Give an example of exactly three unit fractions that when you add them together, you get 1.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$$

What about three unit fractions that add up to 1 and all of their denominators are different from each other.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$$

Part (c) Give a list of ten different unit fractions so that there is no way to take some of them, add those up, and get 1.

For instance, this list does not work. Can you say why?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11}$$

This list doesn't work because I can add together $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$ and I can get a sum of 1. I can create a list of ten unit fractions which are way too small to add up to 1, even if you add them all up together... So I think the following will work:

$$\frac{1}{100} \quad \frac{1}{101} \quad \frac{1}{102} \quad \frac{1}{103} \quad \frac{1}{104} \quad \frac{1}{105} \quad \frac{1}{106} \quad \frac{1}{107} \quad \frac{1}{108} \quad \frac{1}{109}$$

However I suspect that if I have the denominators all be different prime numbers, they can never add up to 1 also...I think my argument for showing this is true for two prime numbers will generalize. Let's do a proof by contradiction and assume that the sum of two distinct primes will add to 1:

$$\frac{1}{p_1}+\frac{1}{p_2}=1$$

$$\frac{p_2}{p_1p_2}+\frac{p_1}{p_1p_2}=1$$

$$p_1+p_2=p_1p_2$$

$$p_2=p_1p_2-p_1$$

$$p_2=p_1(p_2-1)$$

Contradiction! We started out saying that p_2 is a prime number but we have just shown that is a multiple of a prime number, p_1, hence it is not prime! I think the only cases we have to consider is if p_2 were 1 or 2, which would make p_2 not be a multiple of p_1.

If $p_2=1$, then looking at the original sum, we could never get a p_1 that would work. If $p_2=2$, then p_1 would have to equal 2, but we stated the two primes were distinct.

I'm assuming a similar argument would work for a sum of three, four, five, etc., distinct primes! We'd just end up with something like:

$$p_2p_3p_4...p_n=p_1(p_2p_3p_4...p_n-p_3p_4p_5...p_n-p_2p_3p_5...p_n-...-p_2p_3p_4...p_{n-1})$$

And similarly we can't have $p_2p_3p_4...p_n$ be a multiple of p_1

Part (d) Look at the list that you made in question (c). There are infinitely many different unit fractions that are not on your list. Can you take some of these "leftover" unit fractions and add them up to get 1? In your sum, you can only use each unit fraction once.

Here's the list I made in question (c)

$$\frac{1}{100} \quad \frac{1}{101} \quad \frac{1}{102} \quad \frac{1}{103} \quad \frac{1}{104} \quad \frac{1}{105} \quad \frac{1}{106} \quad \frac{1}{107} \quad \frac{1}{108} \quad \frac{1}{109}$$

I suppose since the standard answer I gave for many of the above problems involves fractions that are not on my list, so that answer works for this part! So again: $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

Part (e) Can you find three unit fractions that add up to $\frac{4}{5}$? What about $\frac{4}{9}$? $\frac{4}{15}$?

Here's my thinking for $\frac{4}{5}$:

$$\frac{4}{5} = \frac{1}{5} + \frac{3}{5} = \frac{1}{5} + \frac{6}{10} = \frac{1}{5} + \frac{1}{10} + \frac{5}{10} = \frac{1}{5} + \frac{1}{10} + \frac{1}{2}$$

I tried a similar strategy for $\frac{4}{9}$. But when I was working things out on paper in a similar way, I wasn't getting any nice pattern. So tried some more and came up with:

$$\frac{4}{9} = \frac{1}{4} + \frac{7}{36} = \frac{1}{4} + \frac{1}{36} + \frac{6}{36} = \frac{1}{4} + \frac{1}{36} + \frac{1}{6}$$

Again, I tried a similar thing for $\frac{4}{15}$ and had to resort to some guessing and checking. I did eventually find:

$$\frac{4}{15} = \frac{1}{15} + \frac{1}{30} + \frac{1}{6}$$

Some notes about my thinking. First, I realize that the instructions for part (e) don't say anything about the fractions being all different. I think I self-imposed that upon myself. Also, I realized that once I have the solution to $\frac{4}{5}$, I can get unit fractions for $\frac{4}{15}$ by multiplying my answers by $\frac{1}{3}$. However, at the moment, nothing else is jumping out at me.

Part (f) If you add up all of the unit fractions from $\frac{1}{2}$ to $\frac{1}{n}$, when is the result a whole number? For instance, when n is 5, the sum is not a whole number:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$$

My initial suspicion is that it will never equal a whole number. I do know that this sum, the harmonic series, will eventually go off to infinity. So by adding more and more, smaller and smaller, numbers, there are tons of chances for the sum to eventually be a whole number. However, based on my thinking with prime numbers, I think the fact that we will be adding a bunch of reciprocals of primes as we continue on in the sum suggests that we won't ever exactly sum to a whole number along the way. It's intuition, at this point, though.

I also remembered this identity for summing $\frac{1}{n^s}$, which I had to look up for recollection... The Golden Key

$$\Sigma_{n=1}^{\infty}\frac{1}{n} = \Pi_{n=1}^{\infty}(1-\frac{1}{p_n})$$

But then I decided that this isn't going to be useful since we're summing with a finite number of fractions, not infinite.

Returning to Part (e) Since I got stuck on Part (f), and since I didn't really love my approach for Part (e), I decided to go back to Part (e). As I was playing around, I realized that one important identity was:

$$\frac{1}{a} = \frac{1}{a+1} + \frac{1}{a(a+1)}$$

That is an easy algebraic expression to prove (by simplifying the right hand side by getting a common denominator). And to use it, for example, if we wanted to find $\frac{1}{17}$, we can actually use the identity to see it is equal to $\frac{1}{18} + \frac{1}{306}$. And if we wanted to find *three* unit fractions that sum to 1/17, we can apply the same identity to the 1/18 or the 1/306. So we could start with:

$$\frac{1}{17} = \frac{1}{18} + \frac{1}{306}$$

and then get

$$\frac{1}{17} = \frac{1}{18} + \frac{1}{307} + \frac{1}{93942}$$

or

$$\frac{1}{17} = \frac{1}{19} + \frac{1}{342} + \frac{1}{306}$$

Although this isn't helping me better understand Part (e), what it shows me is that you can have infinite unique ways to find a unit fraction as a sum of other unit fractions. And you can keep using this identity to get smaller and smaller and more and more terms.

Returning to Part (f) I'm still struggling with this proof. One thought I had was using proof by induction to show that each time you add a fraction in the harmonic series, you will end up with a simplified sum which can never be an integer. So show it true for a base case, assume it's true for a sum of 1/2+1/3+...+1/n, and then use that result to show that 1/2+1/3+...+1/n+1/(n+1) cannot be an integer. But I kept on having trouble figuring out what to use for the assumption... that the sum is a simplified fraction p/q, where p is not a multiple of q? where p and q are relatively prime? where p is odd and q is even? I started confusing myself.

Part (g) Looking over your work on adding fractions, what questions do you have? Is there anything you are wondering about, or surprised by? How are these problems similar or different from other fraction problems that you have worked on before?

Some things that come to mind:

- Why do we care about unit fractions?
- Why are we concerned with finding *three* unique fractions that sum?
- What are we supposed to get out of parts (d) and (e)? Was the choice of the fractions in part (e) random, or was there some connection/scaffolding that those particular fractions were intended to lead us to?
- How do we prove part (f)? Can it be done by induction?
- Can every fraction be written as a sum of unique unit fractions?
- Can we come up with an algorithm to find an infinite sum of unit fractions to best and quickly approximate irrational numbers like $\sqrt{2}$ or e or π?
- I was surprised that I could find an easy way to showcase any unit fraction as a sum of as many other unit fractions as I wanted.
- I'm not right now seeing a bigger picture for these problems. I keep seeing smaller patterns but right now things aren't quite mushing up into a cohesive whole. Which I'm totally happy with! I'm just getting interested to see how everything will connect together!

Notes on My Reading Selection

What section of the book did you read?

I chose to read Section 8.7: "A Negative Answer to Littlewood's Question about Zeros of Cosine Polynomials.

In a few sentences, try describing the main ideas of the section.

The article begins by introducing the reader to the periodic nature of the cosine graph, and made note that if you have all the solutions to a periodic function given within a whole period, you can then use those solutions to extrapolate all the infinite other solutions

Then the author introduced the most basic form of "the cosine equation" which was $\cos(n_1 x)+\cos(n_2 x) = 0$, where the constants n_1 and n_2 were different. And they argued, using a basic trigonometric identity, that this cosine equation will have a certain number of solutions based on the constants.

The author then uses this simple example of an equation with two terms to move up to one additional term in the equation, before setting up the final problem. The three term equation is here: $\cos(n_1 x) + \cos(n_2 x) + \cos(n_3 x) = 0$ but now we're restricting the coefficients in front of the x term to be distinct and non-negative. We are told (without proof) that it has been shown that this equation will always have *at least* two solutions on the interval $[0, 2\pi)$.

Now we get to Littlewood's question... He said for: $\cos(n_1 x) + \cos(n_2 x) + ... + \cos(n_N x) = 0$ where the coefficients are distinct integers, what are the *minimal* number of solutions that exist on the interval $[0, 2\pi)$. His conjecture? "Possibly $N - 1$, or not much less.

It turns out that this conjecture is good for $N = 1, 2, 3, ..., 10$ because you're *guaranteed* to have *at least* $N - 1$ solutions on the interval. Hence that is the minimal number of solutions! But for $N = 11$, it has been shown that you can find coefficients which yield only 8 solutions on the interval, and for $N = 16$ you can find coefficients which yield only 14 solutions on the interval. In fact, with a computer, a mathematician showed for $N = 140$, you can get only 52 solutions! It seems Littlewood's conjecture was wrong!

Now for the tricky part... Theorem 8.7. It turns out that the minimal number of solutions is going to be way less than $N - 1$ for infinitely many values of N. You can find coefficients such that the number of solutions has an upper bound of $CN^{5/6}\ln(N)$, where C is just a constant that exists. But this value can be shown to be much less than $N - 1$, and the quotient of $CN^{5/6}\ln(N)$ and $N - 1$ goes to 0 as N goes to infinity. What this means is that for some large value of N, we are going to get so so so many fewer solutions than Littlewood expected, practically none when compared to the conjecture of $N - 1$.

What is a new word or term that you learned in your reading? What does it mean?

Interestingly, I don't think there was a word or term that I didn't know in the text. However I got very stuck when trying to interpret the theorem at the end, Theorem 8.7, even though I understood all the words in it.

Describe a place where you got stuck in reading. Then, what steps did you take to try to get unstuck?

I initially got stuck understanding the number of solutions when the cosine equation with only two terms had a certain number of solutions. I understood it algebraically, but I wanted to see what was happening graphically, so I went to Desmos and graphed various examples. I found that for two terms, $\cos(0x) + \cos(1x) = 0$ had 1 solution on the interval $[0, 2pi]$. And I found that for three terms, $\cos(0x) + \cos(1x) + \cos(3x) = 0$ had 2 solutions on the same interval. But then I found $\cos(0x) + \cos(1x) + \cos(3x) + \cos(5x) = 0$ which only has 2 solutions on the same interval, but according to the paper, it should have at least 3 solutions. What I've realized by all this graphing is that I'm clearly missing something when trying to understand the conjecture.

The text also kept on switching between integers and non-negative integers when talking about various iterations of the problem, and that got frustrating.

Lastly, I got confused between the connection in Theorem 8.7 between the sequence with capital Ns and the actual expansion of the cosine sum, with the subscript of little n. I still don't totally understand that. I also don't get if for every N, that inequality $CN^{5/6}\ln(N)$ holds, or just that there are a lot of N values where it does.

What in the reading "clicked" for you? How did the math you read about connect with math you already knew? What did the reading make you wonder about?

I loved that my understanding of periodicity, and the cosine function, made sense to me. I enjoyed seeing a trig identity I knew from years ago pop up in the reading. I am curious what Littlewood was working on that had him think about this particular question involving a sum of cosines with various periods.