21st Century Mathematics with Justin Lanier - Week 2

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I'm working on these problems not having read the chapter in the book.

Part (a)

Point A is (3,2). Point B is (13,9) The midpoint M is at the average of the x- and y-coordinates, so $(\frac{3+13}{2}, \frac{2+9}{2}$. This puts M at (8,5.5).

To get all points C $(c_x, c_y)$and D (d_x, d_y) which have M as the midpoint:

$$(\frac{c_x + d_x}{2}, \frac{c_y + d_y}{2}) = (8, 5.5)$$

So let $C = (c_x, c_y)$, that means $D = (16 - c_x, 11 - c_y)$

Part (b)

Here is an image showing my answer.

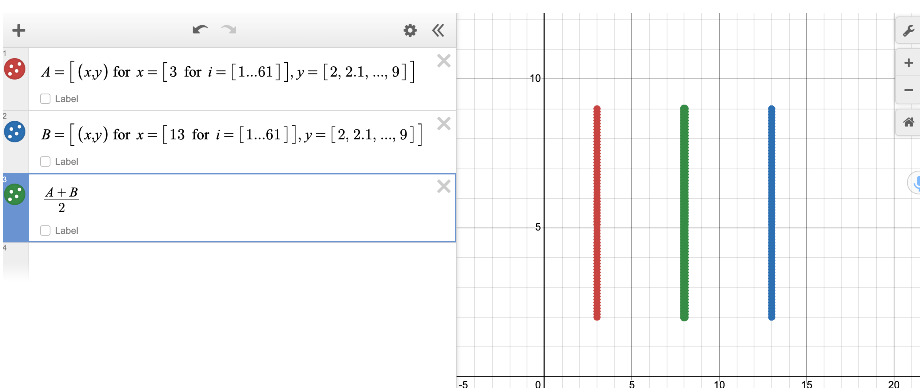


Figure 1: The green points are the midpoints of the red and blue line segments

Part (c)

Here is an image showing my answer. Notice that two line segments (1 dimensional) yield a 2 dimensional ”midpoint region.”

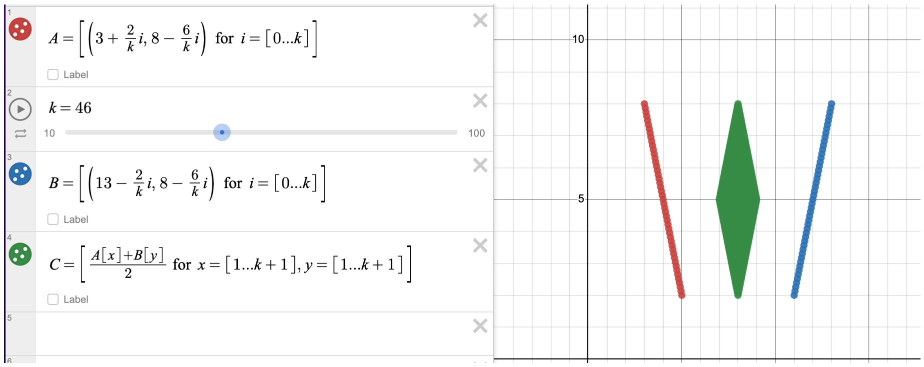
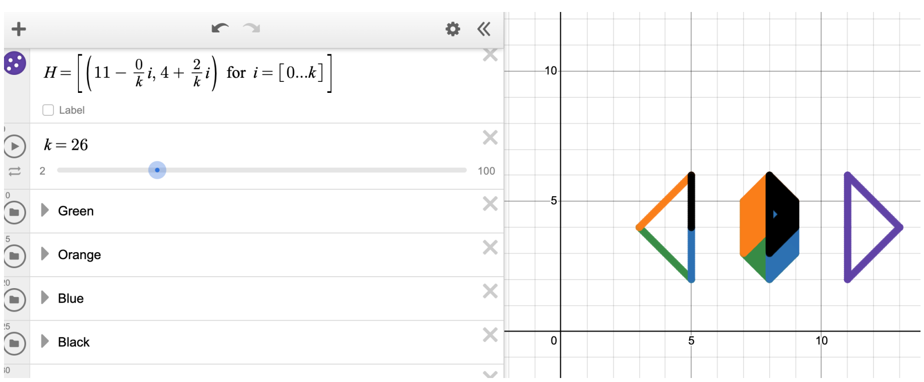


Figure 2: All midpoints for two line segments

Part (d)

The Desmos Calculator page I was using is here:
https://www.desmos.com/calculator/tokuntlegm
If you mess around with the page, you can see how the orange segment of the left hand triangle creates a particular 2D region with the purple triangle. This page is only finding midpoints from the boundaries of the two triangles. But again, those boundaries form a midpoint region that is 2D!



Let’s look specifically at how the orange line on the left generates the orange region in the midpoint shape.

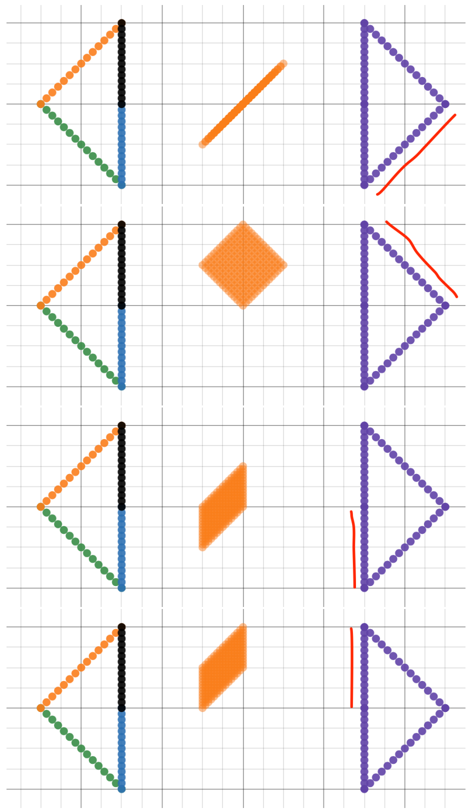


Figure 3: Graphs of the midpoints of the orange segments with the highlighted (underscored red) segments on the right

Look at what we see... All these figures concatenated gets us:

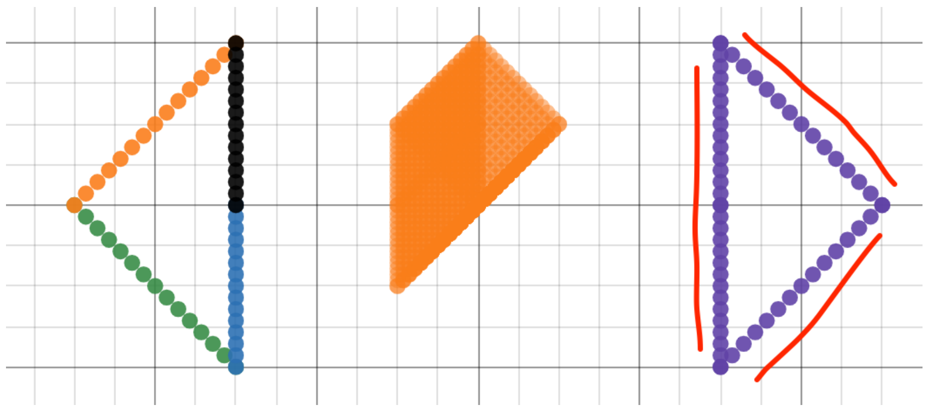


Figure 4: All midpoints connecting the orange line segment to the purple triangle

Part (e)

Let's see what we know! First off, most importantly, it shouldn't matter for the Minkowski symmetrization of a shape if the line of reflection is vertical, horizontal, or at a slant. We're arbitrarily putting this problem on a coordinate grid and drawing a line (such as with the second example given in part (e)). But we can just rotate the diagram so that the line of reflection is facing in any direction we want – and we can ignore the coordinate grid. This is geometry! So let's only worry about vertical lines of reflections – if we have a different line of reflection, we can just rotate the paper to get a vertical line of reflection.

In part (b) we saw that with a vertical line of reflection and two vertical lines which are reflections of each other, we get a single vertical line on the line of reflection that is the Minkowski symmetrization.

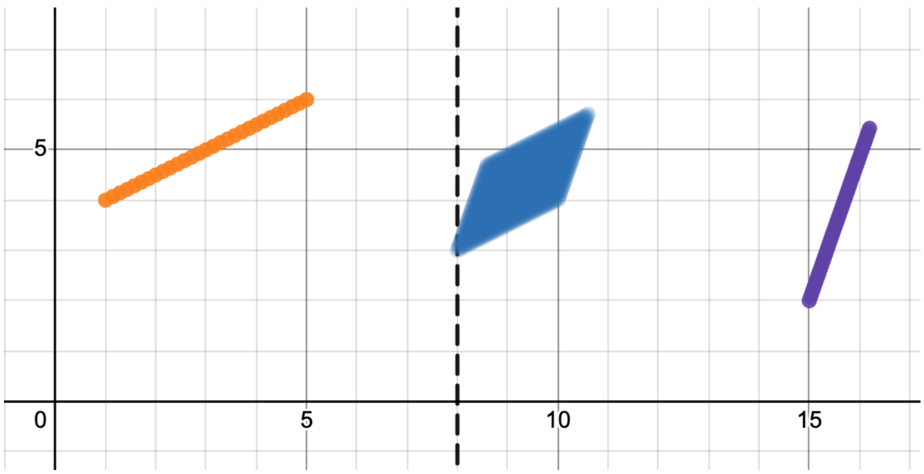
In part (c) we saw that if we slightly tilted the lines (still with a vertical line of reflection), we get a rhombus.

If I were to create two horizontal lines (still with a vertical line of reflection), we'd get a horizontal line.

Now in part (d) we started finding the midpoints between line segments that might not be reflections of each other. The segments could be different lengths! What do we see? They all form parallelograms (some, flattened, forming a line segment).

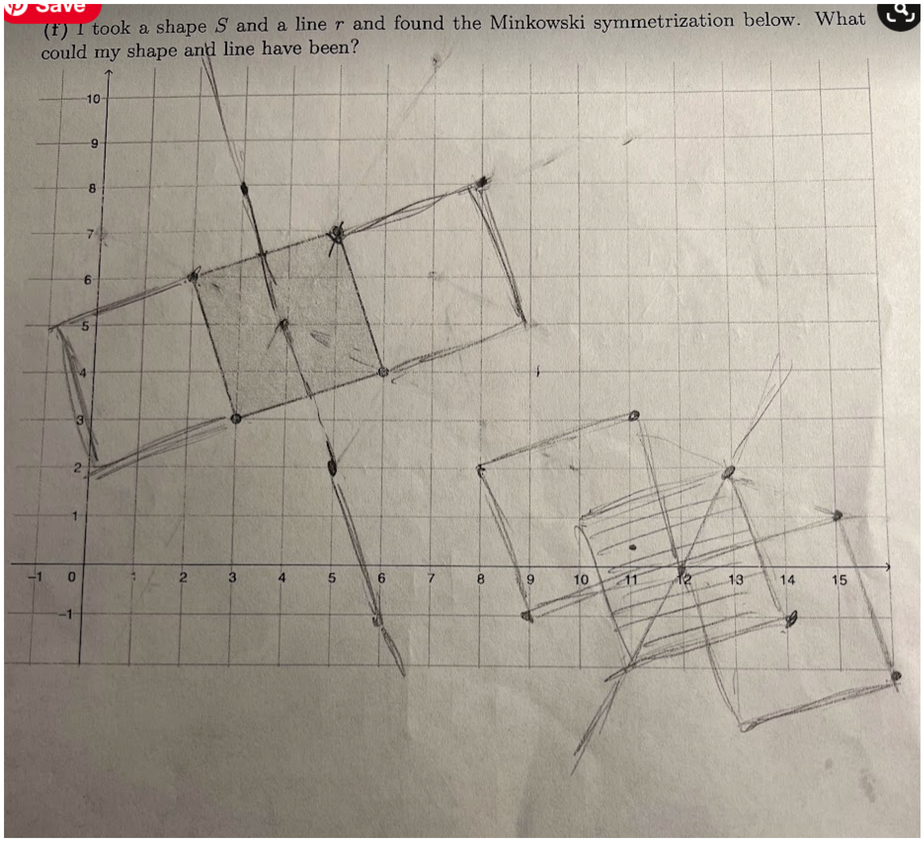
I suspect one way to easily generate the Minkowski symmetrizaton of a polygonal shape is to come up with a quick way to find the set of midpoints between any two line segments (so all the parallelograms). And then concatenate them.

In terms of the two line segments... Here's what I think: If the two segments are perpendicular, the midpoint region will rectangular... If the two segments are parallel, the midpoint region will be a line segment... And if the two segments are at a particular angle to each other, the parallelogram formed will also have that same angle! And in fact, the sides of the parallelogram will also be parallel to the two segments.



Part (f)

Here's what I came up with:



Part (g) Question 1

I'm going to share the conclusion first. I believe all the midpoints of the chords that go through point P will form a circle. More specifically, a circle that goes through point P and also that goes through the center of the original circle.

I can show those two points (P and the center of the original circle) with these illustrations...

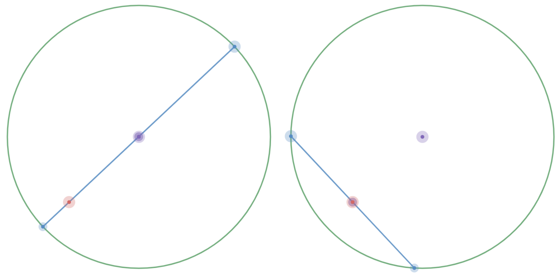


Figure 5: The first image graphs the chord through P and the center of the original circle. And of course the midpoint of that chord is the center of the original circle. The second image graphs the chord through P that is has the midpoint at P. Thus P will be the midpoint of that chord. (This second chord is perpendicular to the first image's chord.

But now let's look at a different chord that goes through P...

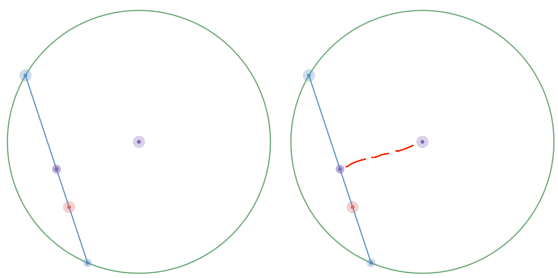


Figure 6: Look! I spy something that looks perpendicular!

If you look at this chord, I see if you connect the midpoint of the chord to the center of the circle, it looks perpendicular. BUT OF COURSE IT IS! The perpendicular bisector of ANY chord in a circle goes through the center of the circle. So it doesn't just look perpendicular. It IS perpendicular. Here are a few more diagrams:

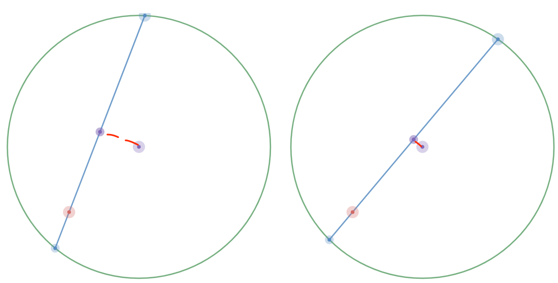
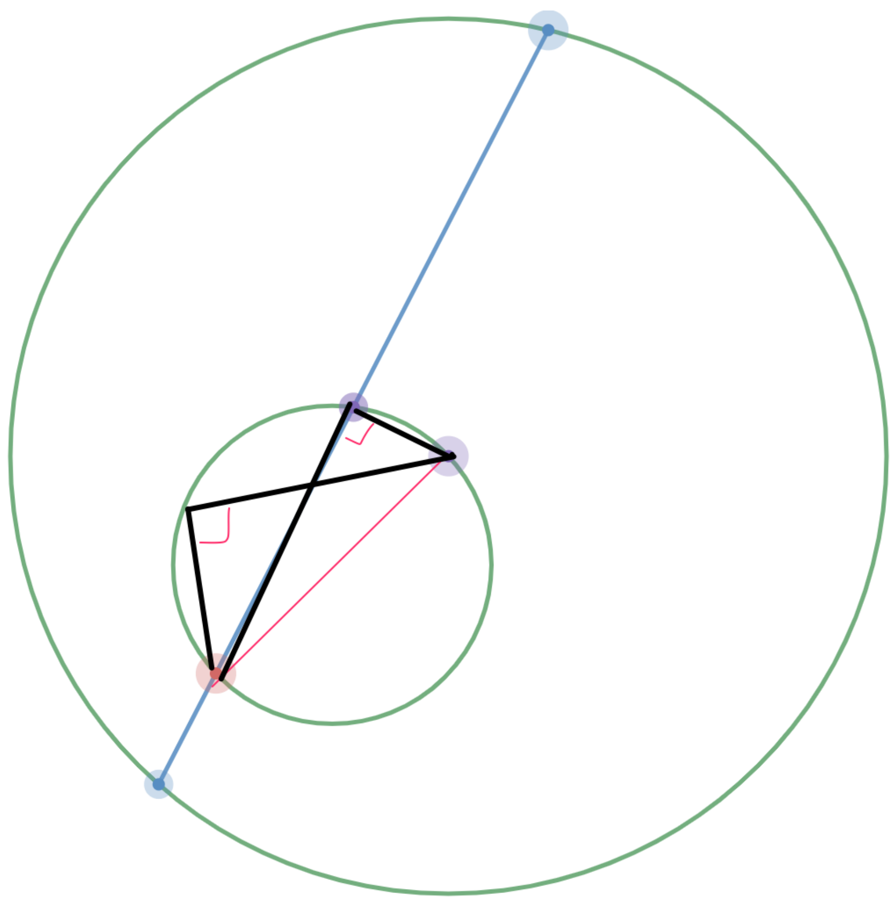


Figure 7: More perpendicularity

Now lets see this circle of points appear! Remember that if you have the diameter of a circle (in this case, the point P and the center of the original circle), then all points on that circle form a right angle to the endpoints of the diameter.



This isn't a full proof. I just realized I assumed the circle and then showed the right angles! And I need to assume the right angles and show the circle results. But I'm tired, so this is as far as I'm going to go with this!

Part (g) Question 2

To work on this, I built a geogebra applet because I first wanted to *see* what would happen.
https://www.geogebra.org/classic/demj2cb9
After seeing the various sets of midpoints that resulted, I decided it wouldn't

be fruitful to engage in all the various algebra involved. But here are some screenshots...

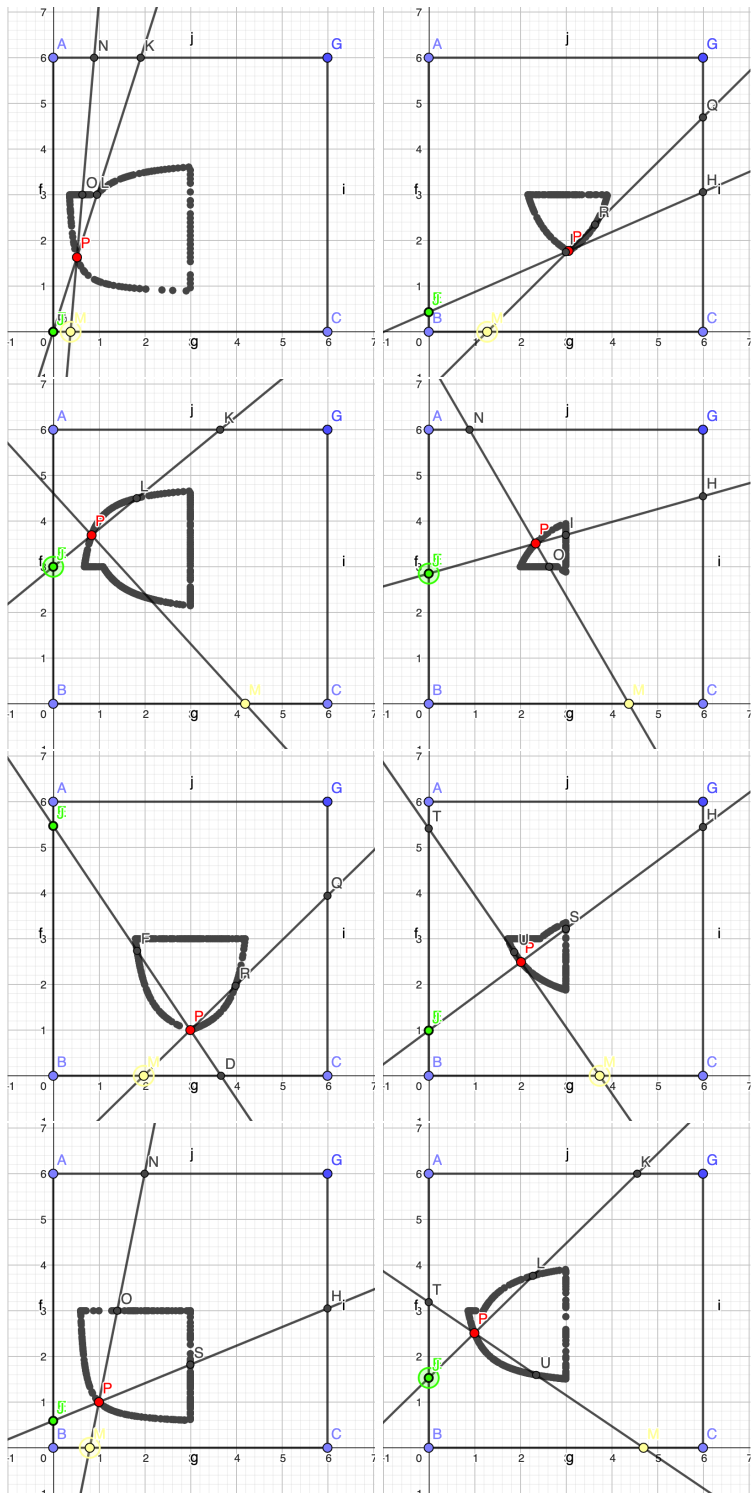
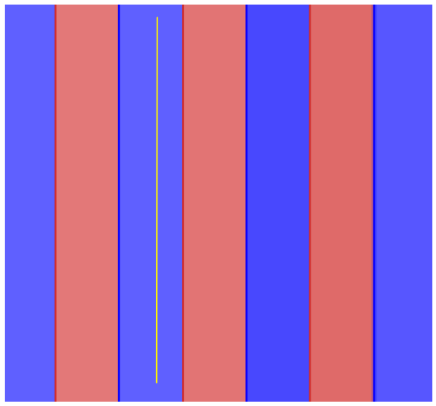
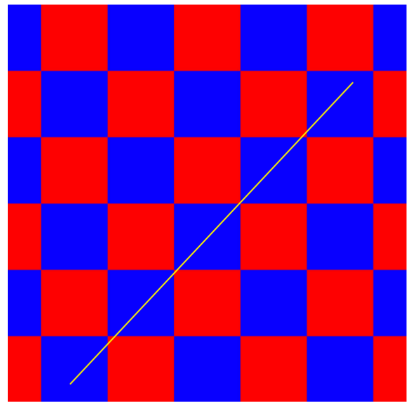


Figure 8: Here are the screenshots for various sets of midpoints of all "chords" that go through a point P in a square.

Part (h)

First look at these diagrams...





If you pick any point on the yellow line drawn, and then look at a distance 1 away on the yellow line, you'll see that the midpoint will also be on the yellow line. And since the yellow line is always on blue, it means the midpoint will also be blue. BAM.

Part (i)

Of the four provided figures, the circle and square seem to satisfy the property that all the interior points are midpoints of pairs of boundary points. And the concave figures don't work. Check out this from Wikipedia:

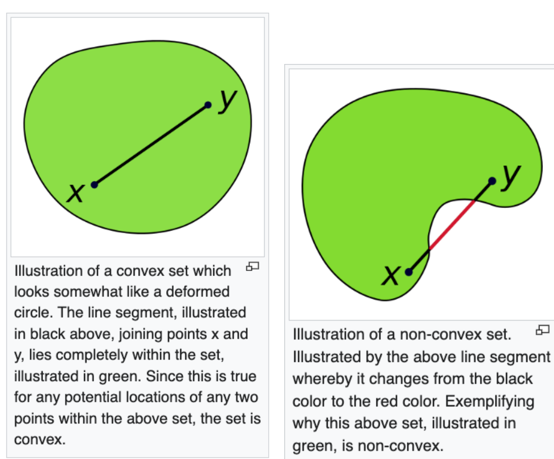


Figure 9: https://en.wikipedia.org/wiki/Convex_set

Part (j)
I don't have any questions because right now my brain is fried. These problems made me think! Finding all sets of points that satisfy a given condition is abstract thinking for me, and I learned my intuition is very off! I needed to be able to play with things, and luckily because I know a bit about geogebra, desmos, and desmos geometry, I could do just that!

Notes on My Reading Selection

What section of the book did you read?

I chose to read Section 10.2: "On divisibility properties of Dyson's rank partition function."

In a few sentences, try describing the main ideas of the section.

The section starts by introducing us to what a partition is. It is the number of unique ways to split up a natural numbers into other natural numbers. So 4 can be expressed as 4, 3+1, 2+1+1, 2+2, 1+1+1+1. There are 5 ways to partition 4. We express this as $p(4) = 5$. It turns out that $p(4) = 5$, $p(9) = 30$, $p(14) = 135$, $p(19) = 490$, and $p(24) = 1575$, and $p(29) = 4565$. Whoa! All of those numbers (4, 9, 14, 19, 24, 29) seem to have a number of partitions that are divisible by 5. Random! It turns out this is true, and is written $p(5n + 4) \equiv 0 \mod 5$.

Eventually the following additional similar relations were found and proved true).

$$p(5n + 4) \equiv 0 \mod 5$$
$$p(7n + 5) \equiv 0 \mod 7$$
$$p(11n + 6) \equiv 0 \mod 11$$
$$p(17303n + 237) \equiv 0 \mod 13$$

We're finding that the partitions of whole infinite arithmetic series of numbers are always divisible by certain primes. At least that's what this suggests, but it took many years just to find these.

But in 2000, Ken Ono proved that for every prime number greater than or equal to 5, we can find a non-trivial infinite arithmetic sequence where all the partitions of those numnbers are evenly divisible by that prime number. Whoa!

Back to the 20th century, 1944. Dyson was looking at $p(5n + 4) \equiv 0 \mod 5$. Let's look at one number in this sequence $p(9) = 30$ to be concrete. It turns out that there is a weird way that you can divide these 30 partitions up into 5 equal size buckets. To do this, we calculate the rank of each partition, which is the largest number in the partition minus the number of numbers in the partition. And then we take the mod 5 of those ranks. That's a lot, so I made a spreadsheet to understand:

Parttition	Largest Number in Partition	Number of Numbers in Partition	Rank (Largest # - # of #s)	Rank mod 5	Parttition	Largest Number in Partition	Number of Numbers in Partition	Rank (Largest # - # of #s)	Rank mod 5
9	9	1	8	3	4+2+2+1	4	4	0	0
8+1	8	2	6	1	4+2+1+1+1	4	5	-1	4
7+2	7	2	5	0	4+1+1+1+1+1	4	6	-2	3
7+1+1	7	3	4	4	3+3+3	3	3	0	0
6+3	6	2	4	4	3+3+2+1	3	4	-1	4
6+2+1	6	3	3	3	3+3+1+1+1	3	5	-2	3
6+1+1+1	6	4	2	2	3+2+2+2	3	4	-1	4
5+4	5	2	3	3	3+2+2+1+1	3	5	-2	3
5+3+1	5	3	2	2	3+2+1+1+1+1	3	6	-3	2
5+2+2	5	3	2	2	3+1+1+1+1+1+1	3	7	-4	1
5+2+1+1	5	4	1	1	2+2+2+2+1	2	5	-3	2
5+1+1+1+1	5	5	0	0	2+2+2+1+1+1	2	6	-4	1
4+4+1	4	3	1	1	2+2+1+1+1+1+1	2	7	-5	0
4+3+2	4	3	1	1	1+1+1+1+1+1+1+1	2	8	-6	4
4+3+1+1	4	4	0	0	1+1+1+1+1+1+1+1	1	9	-8	2

You can see that 6 partitions have a rank of 0 (mod 5), 6 partitions have a rank of 1 (mod 5), 6 partitions have a rank of 2 (mod 5), 6 partitions have a rank of 3 (mod 5), and 6 partitions have a rank of 4 (mod 5).

To me, it's a weird calculation, the rank, but it does equally divide up the partitions of 9 into these buckets equal size buckets! It turns out – and it was proven – that this will hold true for any $p(5n + 4)$

. We can divide up those partitions into 5 buckets using the rank of each partition, and those buckets will have the same number of partitions in them. Whoa! And then this was later proven to be true for $p(7n + 5)$ but using 7 buckets. Whoa!

And finally in 2010, Ono and Bringman proved this would hold true for all these special arithmetic sequences that were studied earlier. In fact, they had a stronger proof which went a bit beyond this and could also work with powers of primes and not just primes.

What is a new word or term that you learned in your reading? What does it mean?

I had learned about partitions, but not about ”rank.” The rank of any partition is the difference of the largest number in the partition and the number of summands in the partition

Describe a place where you got stuck in reading. Then, what steps did you take to try to get unstuck?

I wanted to fully understand putting partitions in equally sized buckets, so I created a spreadsheet to see if my understanding matched what the spreadsheet said – and they did!

What in the reading “clicked” for you? How did the math you read about connect with math you already knew? What did the reading make you wonder about?

The reading was written so clearly that I didn’t have many questions. I think the reason is because they used lots and lots of examples, and because I had been introduced to partitions before so it wasn’t a new idea.