21st Century Mathematics with Justin Lanier - Week 3

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Part (a)

The partitions of 5 are:

5
4+1
3+2
3+1+1
2+2+1
2+1+1+1
1+1+1+1+1

There are 7 partitions of 5.

Part (b)

I notice that the number of partions incease as n increases, and it seems to increase at an increasing rate.

I wonder if there is any explicit formula for $p(n)$. If not, I wonder if there is any approximation that gets infinitely better as n increases for $p(n)$. Clearly $p(n)$ can have an upper bound of 2^n (because that's how many "partitions" a number can have if we count ordering... so for example 4+1 is the same as 1+4.

I already read the section, so I know $p(29)$ is divisible by 5, since we read in the section that $p(5n+4)$ is always evenly divisible by 5.

Part (c)

$32 \equiv 2 \mod 5$
$400 \equiv 0 \mod 5$
$19 \equiv 5 \mod 7$

$699 \equiv 6 \mod 7$
$-2 \equiv 3 \mod 5$
$4 \times 5 \times 9 \times 18 \times 104 \equiv 0 \mod 5$
$2^{100} \equiv 1 \mod 5$

To get the last one, I looked for a pattern...

$$1 \equiv 1 \mod 5$$
$$2 \equiv 2 \mod 5$$
$$4 \equiv 4 \mod 5$$
$$8 \equiv 3 \mod 5$$
$$16 \equiv 1 \mod 5$$
$$32 \equiv 2 \mod 5$$
$$64 \equiv 4 \mod 5$$
$$128 \equiv 3 \mod 5$$

We see this pattern repeats every 4. So let's look at where 2^{100} will be on this list... it will be sorted in the same camp as $2^0, 2^4, 2^8, 2^{12}, \ldots$. Hence, $2^{100} \equiv 1 \mod 5$

Part (d)

Calculating the ranks of the partitions of 4.

partition	largest value	pieces	rank
4	4	1	3
3+1	3	2	1
2+2	2	2	0
2+1+1	2	3	-1
1+1+1+1	1	4	-3

An example of a partition so that the rank is 10 would be: 21+1+1+1+1+1+1+1+1+1+1
The largest values is 21 and the number of pieces is 11. Thus the rank is 10.

An example of a partition so that the rank is -10 would be: 1+1+1+1+1+1+1+1+1+1+1
The largest values is 1 and the number of pieces is 11. Thus the rank is -10.

For a partition of the number n, the partition n has the biggest rank with value $n-1$.

For a partition of the number n, the partition $1+1+1+\ldots+1$ has the smallest rank with value $1-n$.

How do I know this? When subtracting two positive numbers to have the largest difference, to maximize the output, you want the first number to be the largest and the second number to be the smallest. Similarly, when subtracting two positive numbers to have the smallest difference, to minimize the output, you want the first number to the the smallest and the second number to be the largest. In both cases above, that's what I've achieved!

Part (e)

In the last problem set, I already did some work with the partitions of 9

Parttition	Largest Number in Partition	Number of Numbers in Partition	Rank (Largest # - # of #s)	Rank mod 5	Parttition	Largest Number in Partition	Number of Numbers in Partition	Rank (Largest # - # of #s)	Rank mod 5
9	9	1	8	3	4+2+2+1	4	4	0	0
8+1	8	2	6	1	4+2+1+1+1	4	5	-1	4
7+2	7	2	5	0	4+1+1+1+1+1	4	6	-2	3
7+1+1	7	3	4	4	3+3+3	3	3	0	0
6+3	6	2	4	4	3+3+2+1	3	4	-1	4
6+2+1	6	3	3	3	3+3+1+1+1	3	5	-2	3
6+1+1+1	6	4	2	2	3+2+2+2	3	4	-1	4
5+4	5	2	3	3	3+2+2+1+1	3	5	-2	3
5+3+1	5	3	2	2	3+2+1+1+1+1	3	6	-3	2
5+2+2	5	3	2	2	3+1+1+1+1+1+1	3	7	-4	1
5+2+1+1	5	4	1	1	2+2+2+2+1	2	5	-3	2
5+1+1+1+1	5	5	0	0	2+2+2+1+1+1	2	6	-4	1
4+4+1	4	3	1	1	2+2+1+1+1+1+1	2	7	-5	0
4+3+2	4	3	1	1	1+1+1+1+1+1+1	2	8	-6	4
4+3+1+1	4	4	0	0	1+1+1+1+1+1+1	1	9	-8	2

A total of 16 partitions have an odd number of pieces.
A total of 8 partitions have pieces of different sizes (so, for example, 7+2 counts but 7+1+1 doesn't count).

Let's look at our partitions of 4. A total of 2 partitions have an odd number of pieces. A total of 2 partitions have pieces of different sizes.

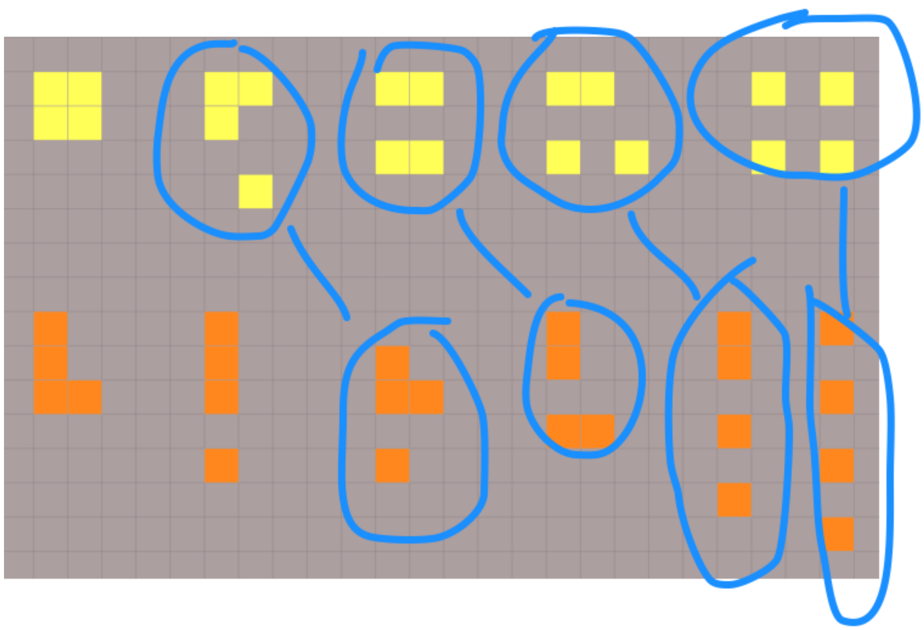
Let's look at our partitions of 5. A total of 4 partitions have an odd number of pieces. A total of 3 partitions have pieces of different sizes.

I used this website [https://www.dcode.fr/partitions-generator] to try to find the answer for $n = 20$. A total of 310 partitions have an odd number of pieces. A total of 64 partitions have pieces of different sizes.

It would seem from this anecdotal evidence that for any given n, that the number of partitions with an odd number of pieces will always be greater than or equal to the number of partitions that have pieces of different sizes, and as the number we're considering gets bigger, the bigger the gap between the two values.

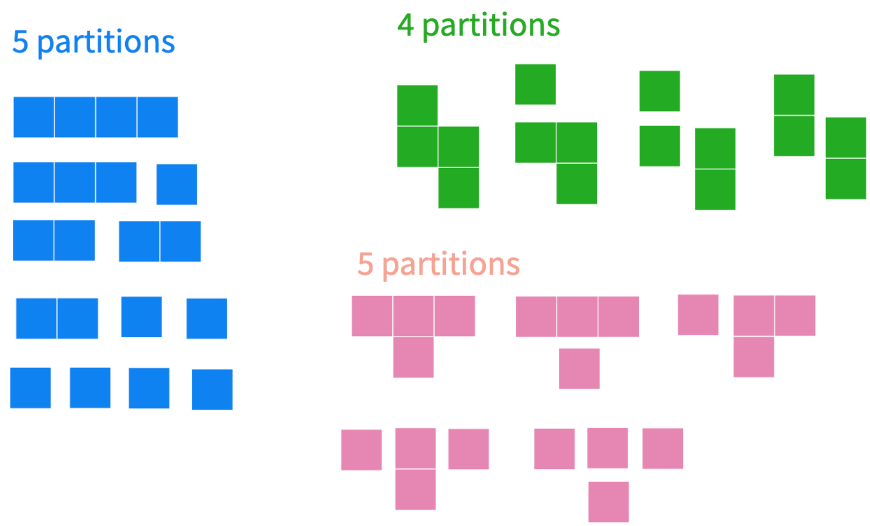
Part (f)

For the first question (how can there be more partitions for the L shape than the square?)... Let's see how the partitions are similar and how they are different...



We can see that most of the partitions are the same, but the original square seems to have some symmetry which the L shape doesn't have. To me, that suggests that the L shape is going to have more partitions.

For the remaining three tetris figures, here are my partitions:



Hm. I'm not sure how to answer the last question (What do you think: for polyominoes made from exactly n squares, what kinds of polyominoes have "a lot" of partitions? What kinds of polyominoes have "very few" partitions?). Based on this single example of tetris shapes, here's a visual I'm thinking about...

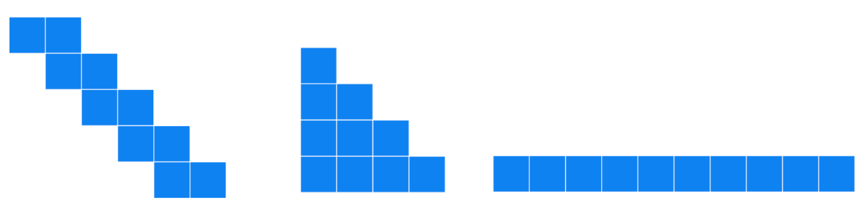


Figure 1 Figure 2 Figure 3

I'm guessing if you design things than can be broken up into a bunch of 1s and 2s, like in figure 1, there aren't many ways to partition it. So I conjecture that a staircase will always give you the smallest way to partition this polyomino.

A "medium" way to partition things will be in figure 3. This is a regular line of polyominos, and this will yield $p(n)$ partitions.

I don't know, but I suspect that getting things as close to a staircase as possible, like in figure 2, might end up being the best, because there are so many ways to create non-congruent miniature shapes.

Part (g)

I have a zillion questions about partitions. Of course, I want to know why they are useful, and why they have been extensively studied. What types of problems does an understanding of partitions come into play? I'd love to know the motivation for creating "rank" of a partition – as it seems fairly arbitrary. Was someone just playing around and stumbled upon it as they were looking at things? I love the polyomino connection and I'm super curious about the answer to the last question about which polyominos have a lot of partitions and which have few! I'm wondering if it's a solved problem and if so, how my conjectures hold up.

Notes on My Reading Selection

What section of the book did you read?

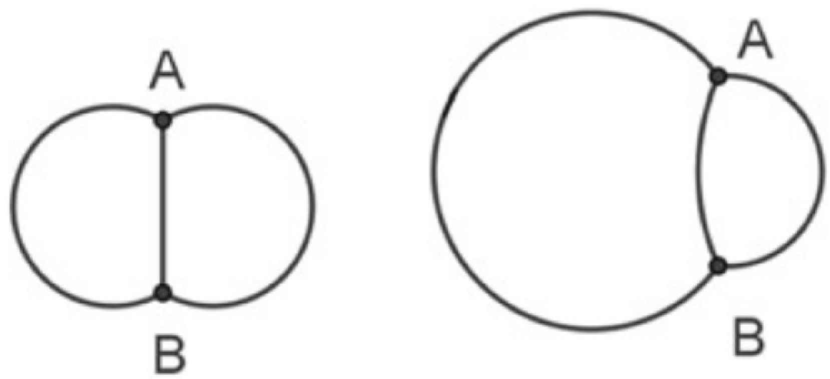
I chose to read Section 2.5: "The Optimality of the Standard Double Bubble"

In a few sentences, try describing the main ideas of the section.

This laid out the history of a question and it's extensions: if you have a fixed area/volume, what's the smallest perimeter/surface area you need to enclose that area/volume. It has been proven, a while ago, that the answer is a circle/sphere.

However, let's say you want TWO regions with the same area to exist, and be connected with a wall of some sort (the wall doesn't have to be straight). Then what can you do? It was shown by undergraduate students in 1993 that the best configuration is two congruent circles that intersect in a particular way and the connecting wall is straight.

If the two regions need to enclose some fixed area but the regions have different areas, then it was proven that two circles of different radii intersect in a particular way, and the connecting wall is curved (a part of a third circle). Below is an illustration from the text.



The 3D version of this problem, where you have two equal volumes that were being connected by a wall, was solved in 1995 by researchers and their approach reduced to calculating 200,260 integrals! Yucko! However in 2002, M. Hutchings, F. Morgan, M. Ritore, and A. Ros, came up with a proof if the volumes were different! And their proof didn't involve using a computer to solve integrals. Their answer was the expected 3D analogue to the 2D problem, with two spheres of different sizes, and the wall connecting them is the piece of a different sized sphere.
What is a new word or term that you learned in your reading? What does it mean?

The term I didn't know was "standard double bubble" but it was defined in the text.

Describe a place where you got stuck in reading. Then, what steps did you take to try to get unstuck?

The article was very general and gave a high-level approach with diagrams and descriptions which outlined the problem. As such, with the examples and explanations provided, I didn't reall get stuck.

What in the reading "clicked" for you? How did the math you read about connect with math you already knew? What did the reading make you wonder about?

I just liked that I head that bubbles can act as "optmizing" machines... and they can solve optimization problems much easier than we can. And this is another example of that. (The first time I learned about this, it was in the context of this problem: https://www.youtube.com/watch?v=dAyDi1aa40E)